



TITLE:

# Triangulations of integral polytopes, examples and problems(Topology of Holomorphic Dynamical Systems and Related Topics)

AUTHOR(S):

KANTOR, Jean-Michel

---

CITATION:

KANTOR, Jean-Michel. Triangulations of integral polytopes, examples and problems(Topology of Holomorphic Dynamical Systems and Related Topics). 数理解析研究所講究録 1996, 955: 133-144

ISSUE DATE:

1996-08

URL:

<http://hdl.handle.net/2433/60412>

RIGHT:

## Triangulations of integral polytopes, examples and problems

Jean-Michel KANTOR

We are interested in polytopes in real space of arbitrary dimension, having vertices with integral coordinates: integral polytopes. The recent increase of interest for the study of these polytopes and their triangulations has various motivations; let us mention the main ones:

- the beautiful theory of toric varieties has built a bridge between algebraic geometry and the combinatorics of these integral polytopes [12]. Triangulations of cones and polytopes occur naturally for example in problems of existence of crepant resolution of singularities [1,5].
- The work of the school of I.M. Gelfand on secondary polytopes gives a new insight on triangulations, with applications to algebraic geometry and group theory [13].
- In statistical physics, random tilings lead to some interesting problems dealing with triangulations of order polytopes [6,29].

With these motivations in mind, we introduce new tools: Generalizations of the Ehrhart polynomial (counting points “modulo congruence”), discrete length between integral points (and studying the geometry associated to it), arithmetic Euler-Poincaré formula which gives, in dimension 3, the Ehrhart polynomial in terms of the f-vector of a minimal triangulation of the polytope (Theorem 7).

Finally, let us mention the results in dimension 2 of the late Peter Greenberg, they led us to the study of “Arithmetical PL-topology” which, we believe with M. Gromov, D. Sullivan, and P. Vogel, has not yet revealed all its beauties. We thank these mathematicians for their interest, and Professor Ito for his kind invitation to the Seminar at R.I.M.S. in October 1995, where part of these results were given.

### I Polytopes; counting integral points; triangulations.

#### I.1.

**Definition 1.** A polytope  $P$  in  $\mathbb{R}^d$  is the convex hull of a finite number of points  $\{A_1, \dots, A_n\}$ .

The set of vertices  $\text{Vert}(P)$  is a subset of  $\{A_1, \dots, A_n\}$ .

The polytope  $P$  is called integral (resp. rational) if the  $A_i$ 's can be chosen in  $\mathbb{Z}^d$  (resp. in  $\mathbb{Q}^d$ ).

**Definition 2.** The polytope  $P$  is said to be *elementary* if

$$\text{Vert}(P) = P \cap \mathbb{Z}^d.$$

These polytopes have also been called “free-lattice polytopes”.

Denote by

$$\mathbf{G}_d = \mathbb{Z}^d \ltimes GL(d, \mathbb{Z})$$

the group of affine unimodular maps (affine linear isomorphisms preserving the lattice  $\mathbb{Z}^d$ ).

**Lemma 1 and definition 3.** Let  $\sigma$  be an integral simplex in  $\mathbb{R}^d$ . The following conditions are equivalent:

1)  $\sigma = g(\sigma_{\text{can}})$ ,

where  $g$  is in  $\mathbf{G}_d$  and  $\sigma_{\text{can}}$  is the basic simplex with vertices the origin and

$$\left\{ A_i = \left( \underbrace{0, \dots, 0}_{i-1}, 1, 0, \dots, 0 \right); i = 1, \dots, d \right\}$$

2) The vertices of  $\sigma$  generate  $\mathbb{Z}^d$ .

If these conditions are satisfied,  $\sigma$  is said to be a *primitive* simplex, and the volume of  $\sigma$  is  $\frac{1}{d!}$ .

### Elementary simplices.

Elementary simplices are well known in dimension up to three (see III.1). They coincide with primitive simplices in dimension 1 and 2. Some partial results are known in dimension 4 [26].

### I.2. The Ehrhart polynomial. Let $P$ be an integral polytope.

**Theorem 1.** [10,3] For any integer  $k$ , let

$$i_P(k) = \text{card}\{kP \cap \mathbb{Z}^d\}.$$

1)  $i_P$  is a polynomial in  $k$  ( $k$  in  $\mathbb{N}$ ),

2) The values of this polynomial at negative  $k$  are given by

$$(1) \quad i_P(-k) = (-1)^m \text{card}(kP^0 \cap \mathbb{Z}^d)$$

where  $m$  is the dimension of  $P$  (dimension of the affine space generated by  $P$ ), and  $P^0$  denotes the relative interior of  $P$ .

The polynomial  $i_P$  is called the *Ehrhart polynomial* of  $P$ .

### Properties of $i_P$ .

The degree of  $i_P$  is the dimension of  $P$ . For example, for a polytope of dimension  $d$

$$i_P(k) = 1 + a_1(P)k + \cdots + a_d(P)k^d$$

where

$$(2) \quad \begin{cases} a_d(P) = V(P), \text{ volume of } P \\ a_{d-1}(P) = \frac{1}{2} \sum V_{d-1}(F) = \frac{1}{2} V_{d-1}(P) \end{cases}$$

summation over all facets  $F$  of  $P$ ,  $V_{d-1}$  denoting the volume of each facet with respect to the lattice induced by  $\mathbb{Z}^d$  on the affine space generated by this facet.

Properties of other coefficients are still mysterious [3.15,17].

### I.3. We introduce new counting functions.

Let  $m$  be an integer and:

$$\Pi_m : \mathbb{R}^d \rightarrow \mathbb{R}^d / m\mathbb{Z}^d$$

the quotient map.

**Definition 4.** For any couple of integers  $m$  and  $k$ , define:

$$(3) \quad i_P(k, m) = \text{card } \Pi_m(kP \cap \mathbb{Z}^d)$$

The functions  $i_P(k, m)$  count integral points in  $kP$  “modulo  $m$ ”, and

$$i_P(k, 0) = i_P(k),$$

is the Ehrhart polynomial.

**Proposition 1.**

$$i_P(k, m) = i_{g(P)}(k, m),$$

for any  $g$  in  $GL(d, \mathbb{Z})$ .

**Proof.**

1) It is enough to consider two cases:

a)  $g(x) = x + a$ ,  $a \in \mathbb{Z}^d$ ,  $Q = P + a$ .

Then

$$\Pi_m(k P \cap \mathbb{Z}^d) = \Pi_m(k P \cap \mathbb{Z}^d) + \Pi_m(k a).$$

b)  $g = A \in GL(d, \mathbb{Z})$ .

$A$  induces a bijection:

$$\bar{A}: \mathbb{R}^d / m\mathbb{Z}^d \rightarrow \mathbb{R}^d / m\mathbb{Z}^d$$

sending  $\Pi_m(k P \cap \mathbb{Z}^d)$  to  $\Pi_m(k A(P) \cap \mathbb{Z}^d)$ .

**Proposition 2.**

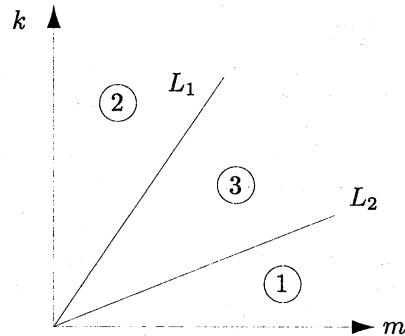
Suppose  $\Pi$  has an integral interior point. Then there exist two rationals  $\alpha$  and  $\beta$  such that:

$$k > m\alpha \implies i_P(k, m) = m^d$$

$$k < m\beta \implies i_P(k, m) = i_P(k)$$

From this one deduces the existence of two critical lines in the plane  $(k, m)$   $L_1$ ,  $L_2$  with the following properties:

in region 1,  $i_P$  is the Ehrhart polynomial  
in region 2,  $i_P$  is  $m^d$   
in region 3,  $i_P$  is unknown

**Proof:**

1) Suppose the interior point is at the origin; let  $\alpha$  be such that

$$\alpha > 0,$$

$$x = (x_i), 0 \leq x_i < \alpha \implies x \in P.$$

If  $(m-1) < k\alpha$ :

$$0 \leq x_i \leq m-1 \implies x \in kP$$

$$kP \cap \mathbb{Z}^d \supseteq [0, m-1]^d$$

and this subset contains all equivalence classes modulo  $m$ .

2) We introduce discrete analogs of euclidean lengths:

**Definition 5**

1) If  $a$  and  $b$  are in  $\mathbb{Z}^d$ , the discrete length between  $a$  and  $b$  is

$$d(a, b) = \text{Card}([a, b] \cap \mathbb{Z}^d) - 1$$

where  $[a, b]$  is the segment joining them.

2) The discrete diameter of an integral polytope is

$$D(P) = \sup_{a,b \in P} d(a,b).$$

The function  $d$  is not a distance!

Now notice that if two integral points  $x$  and  $x'$  satisfy

$$x - x' = mu, \quad u \in \mathbb{Z}^d, \quad u \neq 0$$

then:

$$x' + iu \in [x, x'], \quad i = 0, \dots, m,$$

$$(*) \quad d(x, x') > m.$$

In particular, if

$$m > D(P),$$

(\*) cannot be satisfied for points in  $P$ . Remark now that

$$d(ka, kb) \geq k d(a, b),$$

and deduce that

$$\frac{m}{k} > D(P) \implies i_P(k, m) = i_P(k).$$

**Valuations.** Let  $A$  be any abelian group.

**Definition 6.** A map

$$\varphi : \mathcal{P}_d \rightarrow A$$

is said to be *additive*, or a *valuation* on  $\mathcal{P}_d$ , if whenever  $P, Q, P \cup Q, P \cap Q$  are integral polytopes,

$$\varphi(P \cup Q) + \varphi(P \cap Q) = \varphi(P) + \varphi(Q).$$

The following was proved in [2]:

**Theorem 2.** If  $\varphi$  is any valuation invariant under  $G_d$ , with values in  $A$ , then there exist unique elements  $\alpha_i$  in  $A$  such that

$$\varphi(P) = \sum_{j=0}^d \alpha_j i_P(j)$$

where  $i_P(j)$  are the values of the Ehrhart polynomial of  $P$  at integers  $j$ .

The proof consists in studying the group

$$\Pi = \mathbb{Z}[\mathcal{P}_d] / \sim$$

where  $\mathcal{P}_d$  is the set of all integral polytopes in  $\mathbb{R}^d$ , and  $\Pi$  is the quotient of the free abelian group on  $\mathcal{P}_d$  by the equivalence relation generated by

$$(4) \quad \begin{cases} [P] = (g(P)), \quad g \in G_d \\ [P \cup Q] = [P] + [Q] - [P \cap Q] \quad \text{if } P, Q, P \cup Q \in \mathcal{P}_d. \end{cases}$$

**Remark.** The functions  $i_P(m, k)$  are not additive. Take for example  $d = 1, m = 2$ . For adjacent intervals  $I_1$  and  $I_2$  with at least two points

$$\begin{aligned} i_{I_1}(k, 2) &= i_{I_2}(k, 2) = i_{I_1 \cup I_2}(k, 2) = 2 \\ i_{I_1 \cap I_2}(k, 2) &= 1. \end{aligned}$$

**I.4. Triangulations.** The only triangulations we consider are triangulations by rational or integral simplices (the triangulations are then called rational or integral).

**Definition 7.** A triangulation  $\mathcal{T}$  of the polytope  $P$  is called

*primitive* if all simplices are (integral) primitive simplices  
*minimal* if all simplices are elementary.

It is easy to see that minimal triangulations are minimal with respect to the natural partial order on the set of integral triangulations.

**Definition 8.** If  $\mathcal{T}$  is any triangulation, call  $f$ -vector of  $\mathcal{T}$  the vector  $f = (f_i)$ , where  $f_i$  is the number of simplices of dimension  $i$ .

**Lemma 4.** If the integral polytope  $P$  has a primitive triangulation  $\mathcal{T}$ , the Ehrhart polynomial  $i_P$  is determined by the  $f$ -vector  $f(\mathcal{T})$ , and conversely.

**Proof:**

Consider  $P$  as the disjoint union of the relative interiors of simplices of dimension  $i$  (of number  $f_i$  in dimension  $i$ ), and use the formula:

$$i_m(k) = \frac{(k+1) \dots (k+m)}{m!},$$

then:

$$i_P(k) = \sum f_j (-1)^j i_j(-k).$$

**Proposition 3.** Let  $P$  and  $Q$  be two integral polytopes. The following conditions are equivalent:

- (i)  $P$  and  $Q$  have the same Ehrhart polynomial.
- (ii) There exists a  $k$  such that  $kP$  and  $kQ$  have the same Ehrhart polynomial.
- (iii) For all  $k$ ,  $kP$  and  $kQ$  have the same Ehrhart polynomial.

Moreover, if  $P$  and  $Q$  have primitive triangulations  $\mathcal{T}_P$  and  $\mathcal{T}_Q$  the conditions above are also equivalent to

$$f(\mathcal{T}_P) = f(\mathcal{T}_Q)$$

$\mathcal{T}_P$  and  $\mathcal{T}_Q$  are said to be **numerically equivalent**.

**Proof:**

The following is obvious:

$$i_{kP}(n) = i_P(kn), \quad \text{for all } k \text{ and } n.$$

From this one deduces, using the polynomial character of  $i_P$ :

$$(i) \implies (iii) \implies (ii) \implies (i)$$

**I.5.** Given a polytope  $P$ , it is a difficult question to decide whether there exist primitive triangulations of  $P$ .

Let us remark that the proof of theorem 2 (see [2]) shows that primitive triangulations exist stably, that is if replacing  $P$  by  $P \cup Q$ , for some  $Q$ .

**1) Example:** Order polytopes [29].

If  $\mathcal{O}$  is a finite poset of  $d$  elements (partially ordered set):

$$\mathcal{O} = \{y_1, \dots, y_d\}$$

define  $P(\mathcal{O})$  to be the set of all points in  $\mathbb{R}^d$  such that

$$P(\mathcal{O}) = \left\{ x = (x_1, \dots, x_d); x \in \mathbb{R}^d \left\{ \begin{array}{l} 0 \leq x_i \leq 1, \text{ if } i = 1, \dots, d, \\ x_i \leq x_j \text{ if } y_i > y_j \text{ in } \mathcal{O}. \end{array} \right\} \right\}$$

$P(\mathcal{O})$  is an integral convex polytope of dimension  $d$ , whose vertices correspond to the set  $\mathcal{L}(P, 1)$  of maps  $\sigma : \mathcal{O} \rightarrow \{0, 1\}$  such that

$$y_1 < y_2 \text{ in } \mathcal{O} \implies \sigma(y_1) \geq \sigma(y_2).$$

**Theorem 3.**  $P(\mathcal{O})$  has a canonical primitive triangulation.

The primitive simplices of top dimension are given by the maximal chains:

$$x_{i_1} \geq x_{i_d} \geq \dots \geq x_{i_d}$$

associated with the poset.

See [6] for another construction of triangulations of  $\mathcal{O}(P)$  giving explicitly the number of simplices in all dimensions.

2) The following result is proved in [16]:

**Theorem 6.** For any integral polytope  $P$ , there exists an integer  $k$  such that  $kP$  possesses a primitive triangulation.

Let  $t(P)$  be the minimal integer with this property.

Let us recall [12] that to  $P$  is associated a fan  $\Sigma$  and a toric variety  $X_\Sigma$  equipped with an ample line bundle  $\mathcal{L}$ .

Let  $k_{\min}$  be the minimal integer  $k$  such that  $\mathcal{L}^k$  is very ample.

**Conjecture 1.**  $t(P) = k_{\min}$ .

It was noticed by B. Sturmfels (unpublished) that an example from [11] shows that

$$t(P) \geq \dim P - 1$$

in general.

**1.6. Ehrhart polynomial and triangulations: the main conjecture.** Let  $P$  be an integral polytope in  $\mathbb{R}^d$ . The Ehrhart polynomial  $i_P$  is clearly invariant by the group  $G_d$ .

Let  $\mathcal{G}_d$  be the pseudogroup associated to  $G_d$  and  $P$  and  $Q$  two integral polytopes.

**Definition 7.** A map  $\varphi : P \rightarrow Q$  belongs to  $\mathcal{G}_d$  (or “is locally in  $G_d$ ”) if

- $\varphi$  is a homeomorphism
- there exists a rational triangulation  $\mathcal{T}$  of  $P$  (resp.  $\mathcal{T}'$  of  $Q$ ) such that on the interior of each simplex  $\sigma$  of top dimension of  $\mathcal{T}$ ,  $\varphi$  coincides with an element of  $G_d$ , and

$$\varphi(\sigma) \in \mathcal{T}'.$$

**Proposition 4.** The Ehrhart polynomial is invariant with respect to the pseudogroup  $\mathcal{G}_d$ .

**Proof:**

Let  $\varphi$  be as above. The homeomorphism  $\varphi$  preserves  $\mathbb{Z}^d$ : this is clear for an integral point  $a$  which is interior to a simplex of top dimension, because  $\varphi$  coincides there with an element of  $G_d$ . If  $a$  belongs to a face of such a simplex,  $\varphi(a)$  can be expressed by continuity via an element of  $G_d$  and so is still in  $\mathbb{Z}^d$ .

The same argument applies to the lattices  $\frac{1}{k}\mathbb{Z}^d$ , and shows that they are preserved by  $\varphi$ . This allows to extend  $\varphi$  as

$$\varphi_k : kP \mapsto kQ$$

in a compatible manner with  $\varphi$  and with

$$\varphi_k(kP \cap \mathbb{Z}^d) \subseteq kQ \cap \mathbb{Z}^d.$$

Applying the same argument to the inverse of  $\varphi$  shows that

$$\text{Card}(kP \cap \mathbb{Z}^d) = \text{Card}(kQ \cap \mathbb{Z}^d).$$

In dimension two, Peter Greenberg proved the following [14]:

**Proposition 8.** Let  $P$  and  $Q$  be two integral polytopes in  $\mathbb{R}^2$ . They have the same Ehrhart polynomial if and only if they are equivalent with respect to  $\mathcal{G}_2$ .

The considerations above, and some computations with the counting functions  $i_P(k, m)$ , led us to the following

**Conjecture 2.** Let  $P$  and  $Q$  be integral polytopes in  $\mathbb{R}^d$  such that

$$i_P(k, m) = i_Q(k, m), \quad \text{for all } k \text{ and } m.$$

Then there is a linear unimodular map sending  $P$  to  $Q$ .

**(Main) Conjecture 3.** In dimension three and above, it is **not** true in general that if  $P$  and  $Q$  are integral polytopes with the same Ehrhart polynomial, they are equivalent by the pseudogroup  $\mathcal{G}_d$ .

**Remark.** In view of Proposition 3, this can be considered as a kind of “Arithmetical Hauptvermutung”: the problem is to find  $P$  and  $Q$  with *numerically equivalent* primitive triangulations  $\mathcal{T}$  and  $\mathcal{T}'$ , such that  $\mathcal{T}$  and  $\mathcal{T}'$  cannot be refined to rational triangulations  $\mathcal{T}_1, \mathcal{T}'_1$ , *combinatorially equivalent* and with all simplices of top dimensions having same volumes (for both).

We will study in detail the arithmetical specificity in dimension 3 in the next paragraph.

## II. Hilbert’s third problem for rational polytopes.

**Problem.** Let  $P$  and  $Q$  be two integral (resp. rational) polytopes. Under which condition are they *equidecomposable*? *equicomplementable*?

As in usual scissor xcongruence framework [24],  $P$  and  $Q$  are equidecomposable if there exist subdivisions:

$$P = \bigcup_{i \in I} P_i \quad Q = \bigcup_{i \in I} Q_i$$

with  $P_i$  and  $Q_i$  integral, (resp. rational) polytopes with disjoint interior, such that

$$P_i = g_i(Q_i) \quad , \quad g_i \in G_d \quad i \in I.$$

Equicomplementability is defined in a similar way, allowing addition of other polytopes (loc.cit.). This is completely analogous to the usual framework of scissor congruence in the classical sense, but the group of motions considered is here the group of *unimodular* mappings.

**Proposition 9.**  $P$  and  $Q$  rational polytopes are *equicomplementable* over  $\mathbb{Q}$  if and only if they are *equidecomposable*.

The proof of Zylev [28] adapts without any difficulty to this situation.

**Remark:** Scissor congruence as defined above does not preserve counting points. A continuity condition is needed, as for maps of  $\mathcal{G}_d$ .



### III. The case of dimension 3: Arithmetical Euler-Poincaré formula.

III.1. We will use the classification of elementary simplices modulo  $G_3$  [21,27].

**Proposition 10.** Let  $T_{p,q}$  be the simplex with vertices the origin and the points  $A(1,0,0)$ ,  $B(0,1,0)$ ,  $C(1,p,q)$  in  $\mathbb{R}^3$ , with

$$1 \leq p < q, \quad (p, q) = 1.$$

- 1)  $T_{p,q}$  is an elementary simplex of volume  $q/6$ .
- 2) Any elementary simplex of  $\mathbb{R}^3$  is equivalent to some  $T(p, q)$ ;  
 $T(p, q)$  and  $T(p', q')$  are equivalent if and only if

$$q = q' ; \quad p = \pm p' \pmod{q}.$$

The main point in the proof consists in proving that any elementary simplex  $\sigma$  in dimension 3 has width equal one, where the width is defined by

$$w(\sigma) = \inf_{u \in \mathbb{Z}^{3*}} D[u(\sigma)].$$

**Proposition 11.** If  $\sigma = T(p, q)$  is as above, its Ehrhart polynomial is

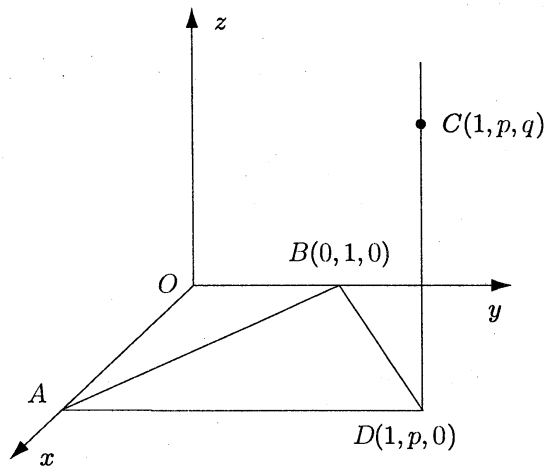
$$i_\sigma(k) = 1 + (2 - q/6)k + k^2 + \frac{q}{6}k^3.$$

**Proof:**

The two coefficients of top degree are easy to compute from the properties of  $i_\sigma(k)$ ; the coefficient  $a_1$  is determined by writing

$$i_\sigma(1) = 4.$$

**Geometric interpretation.** Consider the basic triangle  $OAB$  in  $\mathbb{R}^2$ , and add the point  $D(1, p, 0)$ :



$T(p, q)$  is the pyramid over  $OAB$  with vertex  $C(1, p, q)$ .

Consider the cone  $T'(p, q)$  of vertex  $C$  with basis  $BDA$ . By subdividing the trapeze  $OBDA$  using  $OD$  instead of  $AB$ , one gets two different simplices  $T_1$  and  $T_2$ . It is easy to show that all  $T_1$ ,  $T_2$  and  $T'(p, q)$  are  $G_3$ -equivalent to the simplex

$$T(q) = [0, A, B, E(0, 0, q)].$$

Denoting by the same symbol  $T_1, T_2, T'(p, q)$  by  $T(q)$  one gets

$$T(p, q) \cup T(q) = T(q) \cup T(q)$$

( $\cup$ : union with no common interior points) which implies by additivity (the intersection of the simplices are primitive triangles) that the Ehrhart polynomial of  $T(p, q)$  is equal to the Ehrhart polynomial of  $T(q)$ .

Other remarkable relations between the  $T(p, q)$ 's can be obtained. For example consider the famous decomposition of Euclid of a prism as a union of three simplices [4]. Begin with a simplex  $T(p, q)$  and construct a prism by adding two simplices like in Euclid. One gets

$$\begin{aligned}\mathcal{P} &= I \times \sigma & I &= [0, C(1, p, q)] \\ \sigma &= \{0, A(1, 0, 0), B(0, 1, 0)\}\end{aligned}$$

$$\mathcal{P} = T_{p,q} \cup T' \cup T''$$

where  $T'$  is  $G_3$ -equivalent to  $T_{p,q}$  and  $T''$  is  $G_3$ -equivalent to  $T_{q+1-p,q}$

$$T_2 = T(\beta, q - p) \quad \text{modulo } G_3$$

with

$$\begin{aligned}\alpha q &= 1 \pmod{p} \\ \beta &= q \pmod{q - p}\end{aligned}$$

and  $\sigma$  and  $\sigma'$  are primitive simplices.

**Other decompositions.** Another relation can be obtained by adding a point exterior to  $T(p, q)$  (as in [2]). One gets

$$T(p, q) \cup \sigma = \sigma' \cup T_1 \cup T_2$$

where  $T_1$  and  $T_2$  can be explicitly described.

All these relations suggest that there should be some arithmetical invariants of an elementary triangulation (apart from the sum of volumes of the various simplices).

**III.2. Minimal triangulations in dimension 3.** Let  $\mathcal{T}$  be a minimal triangulation of the polytope  $P$  in  $\mathbb{R}^3$ , and  $f$  the  $f$ -vector of  $\mathcal{T}$ .

**Theorem 7.** (Arithmetic Euler-Poincaré formula).

The Ehrhart polynomial of  $P$  is

$$i_P(k) = 1 + a_1(P)k + \left(\frac{f_2}{2} - f_3\right)k^2 + V k^3$$

where  $V$  is the volume of  $P$  and

$$a_1(P) = f_1 - \frac{3}{2}f_2 + 2f_3 - V.$$

**Proof:**

Results from the proof of lemma 4 and proposition 11 in dimension 3.

**Remarks.** a) The Ehrhart polynomial does not depend on the various volumes of the simplices of dimension 3 which occur in the minimal triangulation.

b) Consider the tetrahedron  $\mathbf{T}$  in  $\mathbb{R}^3$  with vertices the origin and

$$(a, 0, 0); (0, b, 0); (0, 0, c);$$

The only known formula for the number of integral points in  $\mathbf{T}$  involves Dedekind sums; [17.24] here, for small values of the integers  $a, b, c$  theorem 7, allows to compute this number through triangulations.

#### IV. Dimension four and above: Convex triangulations.

**IV.1** The following is proved in [26]:

**Theorem** If  $\sigma$  is an elementary simplex of dimension 4,  $\sigma$  has a primitive facet (face of codimension 1).

This means there exists a basis of  $\mathbb{Z}^4$  such that  $\sigma$  can be written as the convex envelope of the following vectors

$$\begin{pmatrix} 0 & 1 & 0 & 0 & a_1 \\ 0 & 0 & 1 & 0 & a_2 \\ 0 & 0 & 0 & 1 & a_3 \\ 0 & 0 & 0 & 0 & a_4 \end{pmatrix}$$

with  $\text{g.c.d.}(a_1, a_2, a_3, a_4) = 1$

$$0 \leq a_i < a_4 \quad i = 1, \dots, 3.$$

The classification of such simplices is still unknown. In particular:

**Conjecture 4.**

Elementary simplices in dimension 4 have width less or equal to two.

**Definition 8.** A triangulation  $\mathcal{T}$  of the integral polytope  $P$  is said to be *convex* (*projective* in [16]) if the maximal simplices of  $\mathcal{T}$  correspond to the domains of linearity of a convex function on  $P$ .

The set of convex integral triangulations of  $P$  can be identified with a finite set of points in  $\mathbb{R}^N$ , and the secondary polytope  $Q(P)$  is defined as the convex hull of this set. From [13], we know that the *edges* of  $Q(P)$  correspond exactly to elementary transformations (called flips or modifications). We have

**Proposition 13.** Two minimal regular triangulations can be connected by elementary transformations.

This result allows to study problems mentioned above using secondary polytopes. In general elementary transforms of elementary simplices can be elementary or not.

We hope to come back to this.

#### REFERENCES

- [1] BATYREV V.; Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties, *J. Alg. Geom.*, **3** (1994), 493-535.
- [2] BETKE V., KNESER M.; Zerlegungen und Bewertungen von Gitterpolytopen, *J. Crelle*, **358** (1985), 202-208.
- [3] BRION M.; *Polyèdres et réseaux*, L'Enseignement Mathématique, **38** (1992), 71-88.
- [4] BRION M.; *Points entiers dans les polyèdres convexes d'après Pommershein, Morelli, Kantor-Khovanski, ...*, Séminaire Bourbaki, Mars 1994.
- [5] CARTIER P.; *Décomposition des polyèdres: le point sur le troisième problème de Hilbert*, Séminaire Bourbaki, 1984-85, n°646.
- [6] DAIS D.; *Enumerative combinatorics of invariants of certain complex threefolds with trivial canonical bundle*, Bonner Math. Schriften, N°279, Bonn 1995.
- [7] DESTAINVILLE-MOSSERI, BAILLY F.; *Entropy in random tilings: a geometrical analysis in configuration space*, ICQ5, World Scientific, 1995, Proceedings of Avignon Conference.
- [7] DI FRANCESCO, GAUDIN M., ITZYKSON C., LESAGE F.; Laughlin's wave functions, Coulomb gases and expansions of the discriminant, *International Journal of Modern Physics A*, vol.9, **24** (1994), 4257-4351.

- [8] DUPONT J.L., SAH C.H.; Homology of Euclidean groups of motions made discrete and Euclidean scissors congruences, *Acta Mathematica*, **164** (1990), 1-27.
- [9] DUPONT J., KANTOR J.-M.; *Scissor Congruence over  $\mathbb{Q}$* , in preparation.
- [10] EHRHART E.; Polynômes arithmétiques et méthode des polyèdres en combinatoire, *International Series of Numerical Mathematics*, **35**, Birkhäuser 1977.
- [11] EWALD, WESSELS; On the ampleness of invertible sheaves in complete projective toric varieties, *Resultate-Math.*, **19** (1991), no. 3-4, 275-278.
- [12] FULTON W.; *Introduction to toric varieties*, Princeton Univ. Press.
- [13] GELFAND I.M., KAPRANOV, A. ZELEVINSKY; *Discriminants, resultants and multidimensional determinants*, Birkhäuser.
- [14] GREENBERG P.; Piecewise  $SL_2\mathbb{Z}$ -geometry, *Transactions Amer. Math. Soc.* **335**, 2 (Feb. 1993), 705-720.
- [15] KANTOR J.-M., KHOVANSKII A.; Une application du Théorème de Riemann-Roch combinatoire au polynôme d'Ehrhart des polytopes entiers de  $\mathbb{R}^d$ , *C.R. Acad. Sci. Paris*, t.317, I (1993), 501-507.
- [16] G.KEMPF; MUMFORD, SAINT DONAT, *Toroidal Imbeddings*, Springer, LNS 339.
- [17] MORDELL; Lattice points in a tetrahedron and generalized Dedekind sums, *J. Indian Math. (New Series)*, **15**, (1951), 41-46
- [18] MORELLI R.; Translation scissors congruence, *Adv. in Mathematics*, **100**, 1 (July 1993), 1-27.
- [19] MORI S., MORRISON D.R., MORRISON I.; On four-dimensional terminal quotient singularities. *Mathematics of Computation* 51 (1988), no. 184, 769-786.
- [20] MORRISON D.R., STEVENS G.; Terminal quotient singularities in dimensions three and four. *Proceedings of the Amer. Math. Soc.* 90 (1984), no. 1, 15-20.
- [21] ODA T.; *Convex bodies and algebraic geometry*, *Ergebnisse*, Springer-Verlag, 1988.
- [22] POMMERSHEIM J.; Toric varieties, lattice points and Dedekind sums. *Mathematische-Annalen* 295 (1993), no. 1, 1-24.
- [23] RICHTER G., ZIEGLER G.; Universality of polytopes, *B.A.M.S.*, Oct. 1995.
- [24] SAH C.H.; *Hilbert's third problem: scissors congruence*, Research Notes in Mathematics, 33, Pitman P., London, 1979.
- [25] SIEBENMANN L.; Topological manifolds, *Actes du Congrès International des Mathématiciens*, Nice 1960.
- [26] WESSELS U.; Diplomarbeit, Böchum 1989.
- [27] WHITE G.K.; Lattice polyhedra, *Canad. J. Math.*, **16** (1964), 389-396.
- [28] ZYLEV V.B.; Equidecomposability of equicomplementable polyhedra, *Sov. Math. Dokl.*, **161** (1965), 453-455.
- [29] HIBI T.; *Algebraic combinatoric on convex polytopes*. Carslaw Publications. (1992).

**J.-M. Kantor**  
e-mail: [kantor@mathp7.jussieu.fr](mailto:kantor@mathp7.jussieu.fr)

**INSTITUT DE MATHÉMATIQUES**  
**UNIVERSITÉ PIERRE ET MARIE CURIE**  
**Case 247**  
4, Place Jussieu  
F-75525 Paris Cedex 05